ANNALS OF PHYSICS: 68, 172–195 (1971)

Quantum Electrodynamics in Two Dimensions*

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Received July 27, 1970

Operator solutions of the Schwinger model are given in several gauges. The correct interpretation for the covariant solution with indefinite metric is discussed. A class of noncovariant gauges, particularly well suited to the analysis of the gauge-invariant algebra is obtained. As a result of this analysis, it is seen that there are no physical electron excitations in the model. Charge sectors can be introduced, but only at the price of violating the spectrum condition. The Coulomb gauge solution is shown to lead to a very pathological indefinite metric. Quantum electrodynamics as limit of a vector meson theory and broken symmetry aspects are discussed in the concluding sections.

I. INTRODUCTION

Quantum electrodynamics in two dimensions, shown to be exactly soluble by Schwinger [1], has been the subject of various investigations [2, 3, 4]. It is the purpose of the present article to explore a number of interesting features of the model which have received little or no attention in earlier treatments, but which are essential to a correct interpretation of the theory.

The previously overlooked aspects of the Schwinger model will emerge in the course of a formulation of the theory in terms of explicit operator solutions, whose construction will be based on methods used effectively by Klaiber [5] in the case of the Thirring model. Particular emphasis will be placed on the relations between solutions in various gauges and the properties of the gauge-invariant algebra of observables.

Schwinger's original solution will be shown (in Section II) to correspond closely to the Gupta–Bleuler formulation of four-dimensional quantum electrodynamics, in the sense that Maxwell's equations are satisfied only on a physical subspace of the overall (indefinite-metric) "Hilbert space." The Wightman functions of this version of the theory cannot be interpreted directly as physical probability amplitudes. Instead, one is left with the task of extracting the positive-metric Hilbert

* Supported in part by the Banco Nacional do Desenvolvimento Econômico.

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space of physical states and corresponding operator realization of the equations of motion which lie buried in the indefinite-metric formalism. Only then does the striking simplicity of the physical content of quantum electrodynamics in two dimensions become obvious.

As we shall see in Section III, it is possible to construct a one-parametric class of positive-metric operator solutions which are related to that of the Schwinger solution by gauge transformations and which share the properties (a) the Lorentz condition, $\partial_\mu A^\mu = 0$, is satisfied and (b) the electron field satisfies commutation rather than anticommutation relations, a reflection of the bad behavior at infinity of the Coulomb potential in two-dimensional space-time. The positive-metric operator realization derived from Schwinger's covariant solution turns out to be of particular interest, since it contains no spurious (gauge) excitations. With the aid of this solution we shall examine (in Section IV) the algebra of observables and show that the electron excitations completely disappear from the theory. This is in agreement with Schwinger's intuitive picture of the total screening of the electronic charge [1]. The algebra of observables is isomorphic to that of a massive, scalar free field, and the usual picture [6, 7] of charge sectors corresponding to inequivalent representations of the algebra is not valid. We shall see, however, that it is still possible to introduce representations which can be interpreted physically as corresponding to charge sectors, but which violate the spectrum condition.

The Coulomb gauge, which is closely related to the charge sector representations, will be derived (in Section V) as the formal limit of solutions related to our previous ones by gauge transformations. The resulting Wightman functions correspond to the Green's functions of Brown [2] and, as we shall show, define a vector space with a very pathological indefinite metric, a further consequence of the bad asymptotic behavior of the Coulomb potential. The Wightman functions are not tempered distributions, and the energy-momentum spectrum cannot be defined in this gauge.

We shall conclude our study of the Schwinger model with a discussion (in Section VI) of the relation between our solutions and the limit of a vector meson theory [4, 8], and with some comments (in Section VII) on the broken symmetry aspects of the model.

The following notational conventions will be used throughout:

$$x = (x^0, x^1),$$
$$g^{00} = -g^{11} = \epsilon^{10} = \epsilon_{01} = 1,$$
$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^5 = \gamma^0\gamma^1$$
$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \bar{\psi} = \psi^*\gamma^0$$
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

II. Covariant Solution

The covariant solution of quantum electrodynamics in two dimensions with zero electron mass, obtained by Schwinger [1] with functional methods gives rise to the following fermion Wightman functions:

$$\langle 0 \mid \phi(x_1) \cdots \phi(x_n)\, \bar{\phi}(y_1) \cdots \bar{\phi}(y_n) \mid 0\rangle = e^{iF(x,y)} W_0(x_1 \cdots x_n y_1 \cdots y_n) \quad (2.1)$$

with W_0 the free zero mass Wightman function and

$$F(x, y) = \pi \Big\{ \sum_{j<k} [\gamma^5_{x_j}\gamma^5_{x_k}(\Delta^-(x_j - x_k) - D^-(x_j - x_k)) + \gamma^5_{y_j}\gamma^5_{y_k}(\Delta^-(y_j - y_k) - D^-(y_j - y_k))] + \sum_{j,k} \gamma^5_{x_j}\gamma^5_{y_k}(\Delta^-(x_j - y_k) - D^-(x_j - y_k))\Big\}, \quad (2.2)$$

where

$$\Delta^-(x) = \frac{i}{(2\pi)} \int d^2p\, \delta(p^2 - m^2)\, e^{-ip\cdot x}\theta(p_0) \quad (2.3)$$

is the two-point function of a mass m free scalar field and

$$D^-(x) = \frac{i}{(2\pi)} \int d^2p\, \delta(p^2)\, \theta(p_0)[e^{-ip\cdot x} - \theta(\kappa - p_0)] \quad (2.4)$$

is the infrared regularized two-point function of a zero mass field [5].

The Wightman functions (2.1) correspond to renormalized fermion fields differing from Schwinger's unrenormalized ones by a finite renormalization constant (cf. Appendix).

One can represent ϕ in terms of independent free-field operators as

$$\phi(x) = \exp[i(\pi)^{1/2}\, \gamma^5(\tilde{\eta}^+(x) + \tilde{\Sigma}^+(x))]\, \psi(x) \cdot \exp[i(\pi)^{1/2}\, \gamma^5(\tilde{\eta}^-(x) + \tilde{\Sigma}^-(x))], \quad (2.5)$$

where ψ is a free zero mass fermion field, $\tilde{\Sigma} = \tilde{\Sigma}^+ + \tilde{\Sigma}^-$ is a mass m free scalar field with

$$\langle 0 \mid \tilde{\Sigma}(x)\, \tilde{\Sigma}(y) \mid 0\rangle = \frac{1}{i}\, \Delta^-(x - y), \qquad \tilde{\Sigma}^- \mid 0\rangle = 0, \quad (2.6)$$

and $\tilde{\eta} = \tilde{\eta}^+ + \tilde{\eta}^-$ is a zero mass field quantized with indefinite metric

$$\langle 0 \mid \tilde{\eta}(x)\, \tilde{\eta}(y) \mid 0\rangle = \frac{(-1)}{i}\, D^-(x - y). \quad (2.7)$$

It is straightforward to obtain (2.1) from (2.5) using free field commutation relations.

There are two reasons for using an indefinite metric in the quantization of the field $\tilde{\eta}$: (1) it is needed to obtain the minus sign in (2.7), and (2) it is necessary because of the infrared problem of the zero mass scalar field in 2 dimensions [9]. The latter problem could be dealt with by a method invented by Klaiber [5] without the need of indefinite metric, whereas the change in sign in (2.7) intrinsically requires an indefinite metric in the covariant solution. In Section III, where solutions in a positive definite Hilbert space will be obtained we shall employ Klaiber's method.

From the Dirac equation

$$i\gamma^\mu \partial_\mu \phi(x) + \frac{e}{2} \gamma^\mu \lim_{\substack{\epsilon\to 0\\ \epsilon^2<0}} \{A_\mu(x+\epsilon)\, \phi(x) + \phi(x)\, A_\mu(x-\epsilon)\} = 0 \qquad (2.8)$$

one gets, with (2.5),

$$A^\mu(x) = -\frac{\sqrt{\pi}}{e}(\epsilon^{\mu\nu}\partial_\nu \tilde{\Sigma}(x) + \epsilon^{\mu\nu}\partial_\nu \tilde{\eta}(x)) = -\frac{1}{m}(\epsilon^{\mu\nu}\partial_\nu \tilde{\Sigma}(x) + \partial^\mu \eta(x)), \qquad (2.9)$$

where

$$m = e(\pi)^{1/2}, \qquad \epsilon^{\mu\nu}\partial_\nu \tilde{\eta}(x) = \partial^\mu \eta(x).$$

This is in agreement with Schwinger's Green's functions for the "photon" field A^μ which are, up to a (in Schwinger's solution indeterminate) longitudinal zero mass part the Green's functions of a free "vector meson" of mass $e/\sqrt{\pi}$.

Let us look now into the current which corresponds to the properly defined (in analogy to [10]) gauge invariant limit [11]

$$j^\mu(x) = \lim_{\substack{\epsilon\to 0\\ \epsilon^2\neq 0}} [\bar{\phi}(x+\epsilon)\, \gamma^\mu \phi(x) - \langle 0 \mid \bar{\phi}(x+\epsilon)\, \gamma^\mu \phi(x) \mid 0\rangle (1 - ie^\mu A_\mu(x))] f^{-1}(\epsilon), \qquad (2.10)$$

with $f^{-1}(0)$ corresponding to the (in this case, finite) renormalization constant. With (2.5) we get

$$j^\mu(x) = j_f^\mu(x) - \frac{1}{\sqrt{\pi}} \partial^\mu \eta(x) - \frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu}\partial_\nu \tilde{\Sigma}(x), \qquad (2.11)$$

where the first term of the r.h.s. of (2.11) is just the free current

$$j_f^\mu(x) = :\bar{\psi}(x)\, \gamma^\mu \psi(x):. \qquad (2.12)$$

The term

$$j_f^\mu(x) - \frac{1}{\sqrt{\pi}} \partial^\mu \eta(x) \equiv j_L^\mu(x) \qquad (2.13)$$

is a purely longitudinal zero mass contribution to the current. Although $j_L{}^\mu$ creates zero norm states from the vacuum, i.e.,

$$\langle 0 \mid j_L{}^\mu(x)\, j_L{}^\nu(y) \mid 0\rangle = \langle 0 \mid j_f{}^\mu(x)\, j_f{}^\nu(y) \mid 0\rangle + \frac{1}{\pi i}\, \partial^\mu \partial^\nu D^-(x-y) = 0, \tag{2.14}$$

it cannot be set equal to zero, since

$$[j_L{}^\mu(x), \phi(y)] = -\partial^\mu D(x-y)\, \phi(y). \tag{2.15}$$

This means that Maxwell's equations

$$\partial_\nu F^{\mu\nu}(x) = -ej^\mu(x) \tag{2.16}$$

are *not* satisfied, so that the Wightman functions (2.1) do *not* correspond to a solution of (2.8) and (2.16). (This assertion can also be checked directly using Schwinger's own functional methods.)

It is clear, however, that we have a simultaneous solution of (2.8) and the modified equation

$$\partial_\nu F^{\mu\nu}(x) = -e(j^\mu(x) - j_L{}^\mu(x)). \tag{2.17}$$

In analogy to what is done in the Gupta–Bleuler formalism, one can define a physical subspace $\mathscr{H}_{\text{phys}}$ by

$$j_L^{\mu -}(y)\, \Psi = 0, \qquad \forall \Psi \in \mathscr{H}_{\text{phys}}. \tag{2.18}$$

This subspace may be constructed explicitly by applying Wightman polynomials in $F^{\mu\nu}(x)$, $j_L(x)$, and

$$\exp[i \sqrt{\pi}\, \eta^+(x)]\, \phi(x) \exp[i \sqrt{\pi}\, \eta^-(x)]$$

on the vacuum, since $j_L(y)$ commutes with all such quantities for al x and y. Between physical states, (2.17) reduces to (2.16), since

$$\langle \Psi \mid j_L{}^\mu(x) \mid \Phi\rangle = 0, \qquad \forall \Psi, \Phi \in \mathscr{H}_{\text{phys}}. \tag{2.19}$$

The existence of aphysical states and the fact that Schwinger's solution describes quantum electrodynamics only on a subspace arise in a straightforward manner from an examination of the operator solutions (2.5), but are not directly obvious in the original Green's function formulation. We shall come back to this point in Section VI, where quantum electrodynamics will be discussed as a limit of vector meson theory.

III. Noncovariant Solutions

Now that we have a more complete analysis of Schwinger's covariant solution, we turn to the problem of constructing operator realizations of the equations of motion in Hilbert spaces of positive-definite metric. From one's experience with four-dimensional quantum electrodynamics, one does not expect to find explicit covariance and locality of all fields in such formulations. One naturally thinks first of the Coulomb gauge, but, unfortunately, this is not an appropriate framework in two-dimensional space-time, due to the growth at infinity of the Coulomb potential (see Section V). Instead, it is most advantageous to return to the covariant solution of Section II and try to construct an explicit operator solution in the physical subspace.

Let $\mathscr{H}_{\text{phys}}$ be the physical subspace of the previous section, and let $\mathscr{H}_0$ be the space of all null vectors in $\mathscr{H}_{\text{phys}}$. Then in the quotient space $\mathscr{H}_{\text{phys}}/\mathscr{H}_0$ we see that $j_L(x)$ vanishes identically and thus, by definition (2.13) we may identify $\eta(x)$ (in $\mathscr{H}_{\text{phys}}/\mathscr{H}_0$) with the field $j(x)$ defined by

$$j_f^{\mu}(x) = \frac{1}{\sqrt{\pi}}\, \partial^{\mu} j(x). \tag{3.1}$$

By the construction of $\mathscr{H}_{\text{phys}}$, we are led to consider as the appropriate electron field not $\phi(x)$, but rather the field related to $\phi(x)$ by the gauge transformation

$$\begin{aligned} \phi(x) &\rightarrow \exp[i\sqrt{\pi}\, \eta^{+}(x)]\, \phi(x) \exp[i\sqrt{\pi}\, \eta^{-}(x)], \\ A^{\mu}(x) &\rightarrow A^{\mu}(x) + \frac{1}{m}\, \partial^{\mu}\eta(x) = -\frac{1}{m}\, \epsilon^{\mu\nu}\partial_{\nu}\tilde{\Sigma}(x). \end{aligned} \tag{3.2}$$

In this way we are naturally led to a construction very similar to the one employed by Klaiber [5]. We shall adopt both his notation and his infra-red technique, referring the reader to the original article for details.

Consider thus

$$\phi^{\alpha}(x) = \exp\left[i\,\frac{\pi}{2}\,(Q + \tilde{Q}) + i\chi^{+}(x)\right] \psi(x) \exp[i\chi^{-}(x)] \tag{3.3}$$

with

$$\begin{aligned} \chi^{\pm}(x) = {} & \alpha j^{\pm}(x) + \alpha(\sqrt{\pi} - \alpha)\, Q\Delta_{\kappa}^{\pm}(x) + \sqrt{\pi}\,(\sqrt{\pi} - \alpha)\, \tilde{Q}\tilde{\Delta}_{\kappa}^{\pm}(x) \\ & + \gamma^5(\sqrt{\pi}\, \tilde{j}^{\pm}(x) + \sqrt{\pi}\, \tilde{\Sigma}^{\pm}(x)), \end{aligned} \tag{3.4}$$

where $j, \tilde{j}$ are the potential, resp., pseudo potential, of the free current with infrared cut off,

$$\begin{aligned} j(x) &= j^+(x) + j^-(x) \\ &= \frac{1}{\sqrt{2\pi}} \int \frac{dp^1}{\sqrt{2p^0}}\, c^+(p^1)(e^{-ip\cdot x} - \theta(\kappa - p^0)) + \text{h.c.}, \\ \tilde{j}(x) &= \tilde{j}^+(x) + \tilde{j}^-(x) \\ &= \frac{1}{\sqrt{2\pi}} \int \frac{dp^1}{\sqrt{2p^0}}\, \epsilon(p^1)\, c^+(p^1)(e^{-ip\cdot x} - \theta(\kappa - p^0)) + \text{h.c.}, \end{aligned} \tag{3.5}$$

$$j_f{}^\mu(x) = \frac{1}{\sqrt{\pi}}\, \partial^\mu j(x) = \frac{1}{\sqrt{\pi}}\, \epsilon^{\mu\nu}\partial_\nu \tilde{j}(x), \tag{3.6}$$

$$\begin{aligned} \Delta_\kappa{}^{\mp}(x) &= \pm \frac{i}{(2\pi)} \int \frac{dp^1}{2p^0}\, \theta(\kappa - p^0)(e^{\mp ip\cdot x} - 1), \\ \tilde{\Delta}_\kappa{}^{\mp}(x) &= \pm \frac{i}{(2\pi)} \int \frac{dp^1}{2p^0}\, \epsilon(p^1)\, \theta(\kappa - p^0)(e^{\mp ip\cdot x} - 1). \end{aligned} \tag{3.7}$$

Q and $\tilde{Q}$ are the total charge and pseudo charge, respectively.

Apart from the additional massive boson field and a Klein transformation which will simplify matters later on, (3.3) corresponds to a subclass of solutions of the Thirring model [5] with $\beta = \sqrt{\pi}$. This ensures that ϕ^α is an operator in the Fock space of the free fermion and boson fields, and hence the positive definiteness condition is satisfied.

The different values of α correspond to different operator gauges. For $\alpha = \sqrt{\pi}$ we obtain the gauge (3.2).

Introducing (3.3) into the Dirac Eq. (2.8) with $m = e/\sqrt{\pi}$,

$$A^{\alpha\mu}(x) = \frac{1}{m}\left[(\alpha - \sqrt{\pi})\left(j_f{}^\mu(x) - \frac{\alpha}{\sqrt{\pi}}\, \partial^\mu \Delta_\kappa(x)\, Q - \partial^\mu \tilde{\Delta}_\kappa(x)\, \tilde{Q}\right) - \epsilon^{\mu\nu}\partial_\nu \tilde{\Sigma}(x)\right]. \tag{3.8}$$

The gauge invariant current constructed with (3.3) leads to

$$j^\mu(x) = -\frac{1}{\sqrt{\pi}}\, \epsilon^{\mu\nu}\partial_\nu \tilde{\Sigma}(x), \tag{3.9}$$

so its zero mass longitudinal part has been gauged away as expected. With (3.8) and (3.9) we have a solution of Maxwell's equations

$$\partial_\nu F^{\mu\nu}(x) = -ej^\mu(x) \tag{2.16}$$

as an operator identity on the whole Hilbert space.

The fermion Wightman functions can be easily computed from (3.3) as

$$\langle 0 \mid \phi^\alpha(x_1) \cdots \phi^\alpha(x_n)\, \bar{\phi}^\alpha(y_1) \cdots \bar{\phi}^\alpha(y_n) \mid 0\rangle = e^{iF^\alpha(x,y)} W_0(x, y) \qquad (3.10)$$

with

$$\begin{aligned} F^\alpha(x, y) = \sum_{j<k} \{&(\alpha^2 - 2\sqrt{\pi}\,\alpha)\, D^-(x_j - x_k) \\ &+ \pi\gamma^5_{x_j}\gamma^5_{x_k}(\Delta^-(x_j - x_k) - D^-(x_j - x_k)) \\ &- \pi(\gamma^5_{x_j} + \gamma^5_{x_k})\, \tilde{D}^-(x_j - x_k) + (\alpha^2 - 2\sqrt{\pi}\,\alpha)\, D^-(y_j - y_k) \\ &+ \pi\gamma^5_{y_j}\gamma^5_{y_k}(\Delta^-(y_j - y_k) - D^-(y_j - y_k)) \\ &+ \pi(\gamma^5_{y_j} + \gamma^5_{y_k})\, \tilde{D}^-(y_j - y_k)\} \\ &+ \sum_{j,k} \{-(\alpha^2 - 2\sqrt{\pi}\,\alpha)\, D^-(x_j - y_k) \\ &+ \pi\gamma^5_{x_j}\gamma^5_{y_k}(\Delta^-(x_j - y_k) - D^-(x_j - y_k)) \\ &- \pi(\gamma^5_{y_k} - \gamma^5_{x_j})\, \tilde{D}^-(x_j - y_k)\} + \frac{\pi}{2} \sum_{j<k} (1 + \gamma^5_{x_j}\gamma^5_{y_{k-j+1}}), \end{aligned} \qquad (3.11)$$

where D^-, Δ^- are given in (2.4), resp. (2.3), and

$$\tilde{D}^-(x) = \frac{i}{(2\pi)} \int d^2p\, \delta(p^2)\, \theta(p^0)\, \epsilon(p^1)[e^{-ip\cdot x} - \theta(\kappa - p^0)]. \qquad (3.12)$$

One can also write (3.10) as

$$\langle 0 \mid \phi^\alpha(x_1) \cdots \phi^\alpha(x_n)\, \phi^\alpha(y_1) \cdots \phi^\alpha(y_n) \mid 0\rangle = e^{iG(x,y)} W_\alpha(x, y) \qquad (3.13)$$

with

$$\begin{aligned} G(x, y) = \pi \Big\{ &\sum_{j<k} [\gamma^5_{x_j}\gamma^5_{x_k}\Delta^-(x_j - x_k) + \gamma^5_{y_j}\gamma^5_{y_k}\Delta^-(y_j - y_k)] \\ &+ \sum_{j,k} \gamma^5_{x_j}\gamma^5_{y_k}\Delta^-(x_j - y_k) + \sum_{j<k} \tfrac{1}{2}(1 + \gamma^5_{x_j}\gamma^5_{y_{k-j+1}}) \Big\}, \end{aligned} \qquad (3.14)$$

where W_α are the Wightman functions of the Thirring model with $\beta = \sqrt{\pi}$, and the last sum in (3.14) corresponds to the Klein transformation.

Due to the nonlocal character of the gauge transformation (3.1), the fields $\phi^\alpha(x)$ no longer anticommutate for space-like separations, which is a common feature of noncovariant gauges.

In fact, it is easily seen that we have commutation instead:

$$[\phi^\alpha(x), \phi^\alpha(y)]_+ = 2\phi^\alpha(y)\,\phi^\alpha(x), \qquad [\phi^\alpha(x), \phi^\alpha(y)] = 0, \qquad (x-y)^2 < 0. \quad (3.15)$$

To obtain such a simple commutation scheme it was necessary to introduce the Klein transformation (compare with [5]).

On the other hand, it follows from [5] that W_α, and, therefore, our Wightman functions (3.13), are invariant if ϕ^α is taken to transform as a (two-component) scalar field, which may be interpreted as a noncovariant formulation for a "spin $\frac{1}{2}$" field.

IV. The Gauge Invariant Algebra

The whole content of a theory like quantum electrodynamics should already be present in the algebra of gauge invariant quantities. Besides the electromagnetic fields (just the electric field in two dimensions) and the current, this algebra should contain bilocal quantities corresponding to the formally defined

$$T(x, y) \cong \phi(x) \exp\left[ie \int_x^y A^\mu(t)\, dt_\mu\right] \phi^*(y).$$

A natural nonformal definition for such quantities in a theory where ϕ, A^μ are given in terms of free fields (as in (3.3), (3.8), for instance) is:

$$T(x, y) = \exp[iK^+(x, y)]\, \psi(x)\, \psi^*(y) \exp[iK^-(x, y)] \quad (4.1)$$

with

$$K^\pm(x, y) = e \int_x^y A_\mu{}^\pm(t)\, dt^\mu + \chi^\pm(x) - \chi^\pm(y). \quad (4.2)$$

The T so defined enjoys the following properties:

1. It is explicitly gauge invariant under both c-number and q-number free field gauge transformations, as is immediately seen from (4.1) and (4.2).

2. It has the correct locality and Lorentz transformation properties. To see that, we insert $\chi^\pm$, $A_\mu{}^\pm$ given by (3.4), resp. (3.8), into (4.1), (4.2) and, using the commutation relations of the free fields involved, obtain

$$T(x, y) = N(x, y) : \exp\left[-i\frac{e}{m}\int_x^y \epsilon^{\mu\nu}\partial_\nu\tilde{\Sigma}(t)\, dt_\mu\right]: \phi^{\sqrt{\pi}}(x)\, \phi^{*\sqrt{\pi}}(y), \quad (4.3)$$

where

$$N(x-y)$$
$$= \exp -i\pi[D^-(x-y)(1+\gamma_x^5\gamma_y^5) + \hat{D}^-(x-y)(\gamma_x^5+\gamma_y^5) + \tfrac{1}{2}(1-\gamma_x^5\gamma_y^5)] \tag{4.4}$$

and $\phi^{\sqrt{\pi}}$ is the field (3.3) in the gauge $\alpha = \sqrt{\pi}$.

From the commutation relations (3.15) and the locality of the free $\tilde{\Sigma}$ field it follows that T can be associated with a region in space-time corresponding to the path followed in the line integral of (4.2). Also, since under a Lorentz transformation [5]

$$\tilde{D}^-(\Lambda x) = \tilde{D}^-(x) + \frac{i}{2\pi}\chi, \tag{4.5}$$

where χ is the Lorentz angle, and $\phi^{\sqrt{\pi}}$ transforms as a scalar

$$U(\Lambda)\, T(x, y, C)\, U^{-1}(\Lambda) = \exp -\left[\frac{\chi}{2}(\gamma_x^5+\gamma_y^5)\right] T(\Lambda x, \Lambda y, \Lambda C), \tag{4.6}$$

where the dependence on the path C has been explicitly denoted, and T transforms as if it were bilinear in "spin $\frac{1}{2}$" fields.

T has a simple physical interpretation as creating a charge dipole with an electric field between the two charges. The current can be conveniently defined as a limit

$$j^\mu(x) = \lim_{\substack{\epsilon\to 0\\ \epsilon^2\neq 0}} -\operatorname{Tr}\{\gamma^0\gamma^\mu(T(x+\epsilon, x) - \langle 0 \mid T(x+\epsilon, x) \mid 0\rangle)\} \tag{4.7}$$

leading again to

$$j^\mu(x) = -\frac{1}{\sqrt{\pi}}\,\epsilon^{\mu\nu}\partial_\nu\tilde{\Sigma}(x). \tag{3.9}$$

Due to the particularly simple expressions of T in terms of the fields $\phi^{\sqrt{\pi}}$, this gauge will play a central role in the following discussion.

Writing with (3.3),

$$\phi^{\sqrt{\pi}}(x) = \exp[i\sqrt{\pi}\,\tilde{\Sigma}^+(x)]\left(\frac{\kappa e^{-\Gamma'(1)}}{2\pi}\right)^{1/2}\sigma(x)\exp[i\sqrt{\pi}\,\tilde{\Sigma}^-(x)], \tag{4.8}$$

one has from (3.13) and [5]

$$\langle 0 \mid \sigma_1(x_1)\cdots\sigma_1(x_m)\,\sigma_2(x_{m+1})\cdots\sigma_2(x_n)$$
$$\times\,\sigma_1^*(y_1)\cdots\sigma_1^*(y_n)\,\sigma_2^*(y_{m+1})\cdots\sigma_2^*(y_n) \mid 0\rangle = 1, \tag{4.9}$$

$$[\sigma(x), \sigma(y)] = 0, \tag{4.10}$$

and all other Wightman functions not obtainable from (4.9) by permutations are zero.

This leads immediately to

$$\sigma_i(x) = \sigma_i(0) \equiv \sigma_i , \tag{4.11}$$

$$\sigma_i^* \sigma_i = \sigma_i \sigma_i^* = 1. \tag{4.12}$$

The σ's form, therefore, an abelian algebra of constant unitary operators. It is clear that the field algebra is a reducible one (in conformity with the failure of the linked cluster property; cf. Eq. (4.9)) and the Hilbert space cyclically generated from the vacuum $| 0\rangle$ will contain many vacua. There can be conveniently parametrized as

$$| n_1 , n_2\rangle = (\sigma_1)^{n_1} (\sigma_2)^{n_2} | 0\rangle \tag{4.13}$$

with n_1, n_2 arbitrary integers.

One can in standard fashion [12] obtain the irreducible representations of the field algebra as

$$\langle 0 | P | 0\rangle = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \, (\Omega P \Omega)_{\theta_1\theta_2} \tag{4.14}$$

with P an arbitrary Wightman polynomial in $\phi^{\sqrt{\pi}}$ and $A_\mu^{\sqrt{\pi}}$ and $(\Omega P \Omega)_{\theta_1\theta_2}$ defining the irreducible representations

$$(\Omega P \Omega)_{\theta_1\theta_2} \, \delta(\theta_1' - \theta_1) \, \delta(\theta_2' - \theta_2) = \langle \theta_1'\theta_2' | P | \theta_1\theta_2\rangle. \tag{4.15}$$

Here $| \theta_1\theta_2\rangle$ are the "eigenstates" of σ_1, σ_2 :

$$| \theta_1\theta_2\rangle = \frac{1}{2\pi} \sum_{n_1 n_2} e^{i(n_1\theta_1 + n_2\theta_2)} \, | n_1 n_2\rangle. \tag{4.16}$$

In each one of the irreducible sectors, σ_1, σ_2 are c numbers

$$\sigma_1 = e^{i\theta_1}; \qquad \sigma_2 = e^{i\theta_2} \tag{4.17}$$

and the field algebra becomes isomorphic with the algebra of the free scalar field with mass m.

The electrons completely disappear from the theory, in agreement with Schwinger's [1] physical picture of the complete screening of the electric charge, leading to the nonexistence of an electron spectrum.

Although in other gauges the field algebra will no longer possess such a simple structure, it is clear that the observable algebra, being gauge invariant, should have the same representations independent of the gauge of the field algebra it is constructed from. In other words, excitations other than the massive boson ones that will be present in other gauges are not physical and can be gauged away.

Let us look now into gauge invariant (observable) algebra. First we notice that there seem to be two natural candidates for such an algebra:

Case (A). γ^5 invariance is taken as a superselection rule, and the algebra of observables will be generated by the electric field $F^{\mu\nu}(x)$ and the bilocals $T_{11}(x, y)$, $T_{22}(x, y)$.

Case (B). Only gauge invariance is required; then we have a larger observable algebra generated by the $F^{\mu\nu}(x)$ and $T(x, y)$.

In the first case, we have from (4.3), (4.8) and (4.12),

$$T_{11}(x, y) = \left(\frac{\kappa e^{-\Gamma'(1)}}{2\pi}\right) N_{11}(x-y) : \exp i\sqrt{\pi}\left[-\int_x^y \epsilon^{\mu\nu}\partial_\nu\tilde{\Sigma}(t)\, dt_\mu + \tilde{\Sigma}(y) - \tilde{\Sigma}(x)\right]:$$

$$T_{22}(x, y) = \left(\frac{\kappa e^{-\Gamma'(1)}}{2\pi}\right) N_{22}(x-y) : \exp i\sqrt{\pi}\left[-\int_x^y \epsilon^{\mu\nu}\partial_\nu\tilde{\Sigma}(t)\, dt_\mu + \tilde{\Sigma}(x) - \tilde{\Sigma}(y)\right]:,$$

which defines a local isomorphism between the algebra of observables and the algebra of the free mass m scalar field.

There are no inequivalent representations of the observable algebra corresponding to the usual charge sectors [6, 7], since the formal sectors characterized by $| n_1 n_2\rangle$ are equivalent to the vacuum sector

$$\langle 0 | A | 0\rangle = \langle n_1 n_2 | A | n_1 n_2\rangle, \tag{4.19}$$

for any element A of the (A) observable algebra.

One can still look, however, at representations of the (A) algebra that can be physically interpreted as corresponding to charge sectors by considering a limiting situation of a dipole state in which one of the poles is removed to infinity. Since our algebra is isomorphic to the free mass m field algebra, those representations will violate the spectrum condition [13] but are nevertheless worth considering, since the physical situation to which they correspond can, for local measurements, be approximated with arbitrary precision by the dipole states of the vacuum representation [14].

Consider the state

$$| g, x^0\rangle = \exp\left[i\sqrt{\pi}\int dx^1\, g(x^1)\, \partial^0\tilde{\Sigma}(x^0, x^1)\right] | 0\rangle \tag{4.20}$$

where g is any real function with $(-\nabla^2 + m^2)^{1/4} g$ square integrable. The expectation values of the electric field and charge density in this state at time x^0 are given by

$$\langle g, x^0 \mid E(x^0, x^1) \mid g, x^0 \rangle = eg(x^1), \tag{4.21}$$

$$\langle g, x^0 \mid ej^0(x^0, x^1) \mid g, x^0 \rangle = e\,\frac{dg(x^1)}{dx^1}. \tag{4.22}$$

As expected, the expectation value of the total charge in the state $\mid g, x^0\rangle$ vanishes. From (4.22) we see that except for this condition on the total charge, we may construct a state $\mid g, x^0\rangle$ corresponding to any reasonable, localized charge distribution at time x^0. In particular, point charges may be obtained as a limiting situation corresponding to discontinuities in $g(x^1)$.

It is easy to see that the vectors $\mid g\rangle \equiv \mid g, 0\rangle$ (we fix $x^0 = 0$ without essential loss of generality) define unitarily equivalent representations of the algebra of observables (equivalently of the algebra of the free scalar field of mass m). Specifically,

$$\langle g \mid \tilde{\Sigma}_{\text{Fock}}(z_1) \cdots \tilde{\Sigma}_{\text{Fock}}(z_n) \mid g\rangle = \langle 0 \mid \tilde{\Sigma}_g(z_1) \cdots \tilde{\Sigma}_g(z_n) \mid 0\rangle, \tag{4.23}$$

where

$$\begin{aligned}\tilde{\Sigma}_g(z) = T_g^{-1}\tilde{\Sigma}_{\text{Fock}}(z)\, T_g = \tilde{\Sigma}_{\text{Fock}}(z) - \sqrt{\pi} \int dx^1\, g(x^1)\, \partial^0 \Delta(-z^0, z^1 - x^1),\\ T_g = \exp i\sqrt{\pi}\left[\int g(x^1)\, \partial^0 \tilde{\Sigma}_{\text{Fock}}(0, x^1)\, dx^1\right].\end{aligned} \tag{4.24}$$

We now wish to construct representations of the observable algebra which correspond to right-handed, left-handed, and symmetric charge monopole states, starting with appropriate dipole and quadrupole states of the type $\mid g\rangle$. Defining

$$g(x^1, y^1 \mid z^1) = \tfrac{1}{2}\epsilon(z^1 - x^1) - \tfrac{1}{2}\epsilon(z^1 - y^1), \tag{4.25}$$

which by (4.22) corresponds to a positive charge at x^1 and a negative one at y^1, we set, for an arbitrary observable A in the (A) algebra,

$$\begin{aligned}\langle x^1(\pm) \mid A \mid x^1(\pm)\rangle &= \lim_{a\to\pm\infty} \langle g(x^1, a \mid \cdot) \mid A \mid g(x^1, a \mid \cdot)\rangle\\ \langle x^1(s) \mid A \mid x^1(s)\rangle &\\ = \lim_{a\to\infty} \Big\langle \frac{g(x^1, a \mid \cdot)}{2} &+ \frac{g(x^1, -a \mid \cdot)}{2} \,\Big|\, A \,\Big|\, \frac{g(x^1, a \mid \cdot)}{2} + \frac{g(x^1, -a \mid \cdot)}{2} \Big\rangle.\end{aligned} \tag{4.26}$$

Strictly speaking, $\mid g(x^1, a \mid \cdot)\rangle$ is a well defined vector only for step functions with rounded-off corners. However, the expectation values of (4.26) are meaningful even in the point-charge limit.

The charge monopole states which we have written as improper vectors $| x^1(\pm)\rangle$ and $| x^1(s)\rangle$ correspond as in (4.23) and (4.24) to representations

$$\begin{aligned} \tilde{\Sigma}_{x^1(\pm)}(z) &= \tilde{\Sigma}_{\text{Fock}}(z) - \sqrt{\pi} \int_{x^1}^{\pm\infty} \partial^0 \Delta(-z^0, z^1 - y^1)\, dy^1, \\ \tilde{\Sigma}_{x^1(s)}(z) &= \tilde{\Sigma}_{\text{Fock}}(z) - \frac{\sqrt{\pi}}{2} \int \epsilon(y^1 - z^1)\, \partial^0 \Delta(-z^0, z^1 - y^1)\, dy^1. \end{aligned} \tag{4.27}$$

It is clear that these free-field representations are inequivalent to the Fock one and even have different asymptotic behaviour for $z^1 \to \pm\infty$:

$$\underset{z^1 \to \infty}{E(z)} \sim \cos mz^0.$$

From [13] it follows that these representations violate the spectrum condition, and it is easily seen that, in fact, there is no unitary implementation of the translation group in them. Physically, this is quite understandable since the representations (4.27) are based on states which represent a charge localized at x^1 for $x^0 = 0$ and an electric field extending to infinity. The charge remains localized within a finite space region but is not constant in time, due to currents extending to infinity. Also the energy-momentum leaks in and out from infinity, leading to the nonexistence of a unitary representation of the translation group, since the system is not a conservative one.

We can, of course, also obtain representations corresponding to an arbitrary number of monopoles and even continuous charge sectors, corresponding to the fact that the charge is not a conserved quantum number.

In the case of the (B) algebra, the irreducible representations satisfying spectrum condition correspond to the $| \theta_1\theta_2\rangle$ sectors, where T_{11} and T_{22} are given by (4.18) and

$$\begin{aligned} T_{12}(x, y) &= \left(\frac{\kappa e^{-\Gamma'(1)}}{2\pi}\right) N_{12}(x - y) \\ &\quad \times :\exp i\sqrt{\pi}\left[-\int_x^y \epsilon^{\mu\nu}\partial_\nu \tilde{\Sigma}(t)\, dt_\mu - \tilde{\Sigma}(x) - \tilde{\Sigma}(y)\right]: e^{i(\theta_1 - \theta_2)}. \end{aligned} \tag{4.28}$$

There are infinitely many inequivalent representations of the (B) algebra corresponding to the broken γ^5 symmetry angle $(\theta_1 - \theta_2)$. By requiring parity to be a good quantum number, the only possibility is $\theta_1 - \theta_2 = 0$.

One can also introduce, as for the (A) algebra, representations which violate the spectrum condition and correspond to charge sectors.

V. Coulomb Gauge

In Sections III and IV we have presented a positive-metric version of two-dimensional quantum electrodynamics which is particularly well suited to the analysis of the algebra of observables. From the intuitive point of view, however, this treatment leaves something to be desired, since the "electron" field $\phi^{\sqrt{\pi}}$ has the rather strange property of creating only charge zero states from the vacuum. The question arises whether it might not be possible, by means of a gauge transformation (which will not affect the algebra of observables), to construct an electron field which not only satisfies the field equations of QED, but also creates a localized charge one state from the vacuum (such states do not, of course, correspond to vectors in the Hilbert space of the $\phi^{\sqrt{\pi}}$, but, as was shown in Section IV, may be constructed as the limit of vectorial states). We shall discover that the construction of such a charged electron field will lead naturally to the Coulomb-gauge formulation of QED in two dimensions.

The simplest way to approximate an electron field with the desired properties is to define (see (4.20)),

$$\phi_a(x) = \exp i\sqrt{\pi}\left[\int_{x^0=y^0} \tfrac{1}{2}(g(x^1, a \mid y^1) + g(x^1, -a \mid y^1))\, \partial^0\tilde{\Sigma}^+(y)\, dy^1\right] \phi^{\sqrt{\pi}}(x)$$

$$\cdot \exp i\sqrt{\pi}\left[\int_{x^0=y^0} \tfrac{1}{2}(g(x^1, a \mid y^1) + g(x^1, -a \mid y^1))\, \partial^0\tilde{\Sigma}^-(y)\, dy^1\right]. \quad (5.1)$$

The improper state $\phi_a(x) \mid 0\rangle$ has a charge distribution which differs from that of $\mid \tfrac{1}{2}(g(x^1, a \mid 0) + g(x^1, -a \mid 0))\rangle$ of Section IV, (4.20), (4.24) only by a short-range cloud of total charge zero.

The "field" $\phi_a(x)$ satisfies the equation of QED with $A_1 = 0$, and thus corresponds to the Coulomb gauge. Of course, it is only in the limit $a \to \infty$ that we expect to be able to define a translationally invariant charged field $\phi_c(x)$.

To see what sorts of problems might be encountered in passing to the limit, let us examine the two-point function

$$\langle 0 \mid \phi_a(x)\, \phi_a^*(y) \mid 0\rangle = \left(\frac{\kappa e^{-\Gamma'(1)}}{2\pi}\right) \langle 0 \mid \sigma_x \sigma_y^* \mid 0\rangle\, e^{[\chi_a^-(x), \chi_a^+(y)]}, \quad (5.2)$$

where

$$\chi_a^{\pm}(x) = \sqrt{\pi}\, \gamma^5 \tilde{\Sigma}^{\pm}(x) + \sqrt{\pi} \int_{x_0=y_0} \frac{1}{2}(g(x^1, a \mid y^1) + g(x^2, -a \mid y^1))\, \partial^0\tilde{\Sigma}^{\pm}(y)\, dy^1,$$

so that

$$
\begin{aligned}
i[\chi_a^-(x), \chi_a^+(y)] = \frac{\pi}{2} \int dt^1\, \partial_0^2 \Delta^-(x^0 - y^0, t^1) \Big\{ & | x^1 - y^1 - t^1 | \\
& - \frac{1}{2} | x^1 + a - t^1 | - \frac{1}{2} | x^1 - a - t^1 | - \frac{1}{2} | a - y^1 - t^1 | \\
& - \frac{1}{2} | a + y^1 + t^1 | + \frac{1}{4} | 2a - t^1 | + \frac{1}{4} | 2a + t^1 | + \frac{1}{2} | t^1 | \Big\} \\
- \frac{\pi}{2} \int dt^1\, \partial^0 \Delta^-(x^0 - y^0, y^1) \Big\{ & (\gamma_x^5 + \gamma_y^5)\, \epsilon(x^1 - y^1 - t^1) \\
& - \gamma_x^5 \frac{\epsilon(x^1 - a - t^1)}{2} - \gamma_x^5 \frac{\epsilon(x^1 + a - t^1)}{2} \\
& - \gamma_y^5 \frac{\epsilon(a - y^1 - t^1)}{2} + \gamma_y^5 \frac{\epsilon(a + y^1 + t^1)}{2} \Big\} \\
+ \pi \gamma_x^5 \gamma_y^5 \Delta^-(x - y). &
\end{aligned}
\tag{5.3}
$$

Asymptotically, for large a,

$$
\begin{aligned}
[\chi_a^-(x), \chi_a^+(y)] = & - \frac{\pi i}{2} \int dt^1\, \partial_0^2 \Delta^-(x^0 - y^0, t^1) \left(| x^1 - y^1 - t^1 | - a + \frac{1}{2} | t^1 | \right) \\
& + \frac{\pi i}{2} \int dt^1\, \partial^0 \Delta^-(x^0 - y^0, t^1)(\gamma_x^5 + \gamma_y^5)\, \epsilon(x^1 - y^1 - t^1) \\
& - \pi i \gamma_x^5 \gamma_y^5 \Delta^-(x - y) + r(a, x, y),
\end{aligned}
\tag{5.4}
$$

where $r(a, x, y)$ vanishes exponentially for x, y fixed and $a \to \infty$.

The term

$$\frac{i\pi}{2} \int dt^1\, \partial_0^2 \Delta^-(x^0 - y^0, t^1) = \frac{\pi}{4} ma e^{-im(x^0 - y^0)} \tag{5.5}$$

leads to a divergent two-point function (even after smearing with test functions) in the limit. We are thus led to make the additional gauge transformation

$$\hat{\phi}_a(x) = e^{i\zeta_a^+(x^0)} \phi_a(x)\, e^{i\zeta_a^-(x^0)}, \tag{5.6}$$

where

$$\zeta_a^\pm(x^0) = \sqrt{\frac{\pi m a}{4}}\, b^\pm e^{\pm i m x^0} \tag{5.7}$$

and

$$[b^+, b^-] = 1. \tag{5.8}$$

The $b^\pm$ are creation and destruction operators of a harmonic oscillator of angular frequency m, which we have quantized in an indefinite-metric Hilbert space adjoined to the positive-metric space of the $\phi^{\sqrt{\pi}}$ by means of a tensor product. It turns out that the gauge transformation (5.6), which affects neither the Coulomb gauge condition $A_1 = 0$ nor the role of the modified electron field as an approximate charge-raising operator, leads to a well-defined set of Wightman functions in the limit $a \to \infty$. In particular, defining (formally)

$$\phi_c(x) = \lim_{a\to\infty} \exp\left[-\frac{i\pi}{2}(Q + \tilde{Q})\right] \hat{\phi}_a(x), \tag{5.9}$$

where the Klein transformation has been undone in order to compare our result with [2], we obtain

$$\langle 0 \mid \phi_c(x_1) \cdots \phi_c(x_n)\, \phi_c{}^*(y_1) \cdots \phi_c{}^*(y_n) \mid 0\rangle = e^{iF_c(x,y)} W_0(x, y), \tag{5.10}$$

with

$$\begin{aligned} F_c(x, y) = \pi \sum_{j<k} \Big\{ &\left(\frac{\partial_0{}^2}{\nabla^2} \Delta^-(x_j - x_k) - D^-(x_j - x_k)\right) \\ &+ \gamma^5_{x_j}\gamma^5_{x_k}(\Delta^-(x_j - x_k) - D^-(x_j - x_k)) \\ &+ (\gamma^5_{x_j} + \gamma^5_{x_k})\left(\frac{\partial^1\partial^0}{\nabla^2} \Delta^-(x_j - x_k) - \tilde{D}^-(x_j - x_k)\right) \\ &+ \left(\frac{\partial_0{}^2}{\nabla^2} \Delta^-(y_j - y_k) - D^-(y_j - y_k)\right) \\ &+ \gamma^5_{y_j}\gamma^5_{y_k}(\Delta^-(y_j - y_k) - D^-(y_j - y_k)) \\ &+ (\gamma^5_{y_j} + \gamma^5_{y_k})\left(\frac{\partial^1\partial^0}{\nabla^2} \Delta^-(y_j - y_k) - \tilde{D}^-(y_j - y_k)\right)\Big\} \\ &+ \pi \sum_{j,k} \Big\{\left(D^-(x_j - y_k) - \frac{\partial_0{}^2}{\nabla^2} \Delta^-(x_j - y_k)\right) \\ &+ \gamma^5_{x_j}\gamma^5_{x_k}(D^-(x_j - y_k) - \Delta^-(x_j - y_k)) \\ &+ (\gamma^5_{x_j} + \gamma^5_{y_k})\left(\tilde{D}^-(x_j - y_k) - \frac{\partial^1\partial^0}{\nabla^2} \Delta^-(x_j - y_k)\right)\Big\}. \end{aligned} \tag{5.11}$$

In (5.11)

$$\frac{1}{\nabla^2} f(x) = \int dt^1 \left(\frac{1}{2} \mid x^1 - t^1 \mid + \frac{1}{4} \mid t^1 \mid\right) f(x^0, t^1). \tag{5.12}$$

The presence of the second term in the Green's function ∇^{-2} is due to our particular choice of limiting procedure (with two symmetrically placed image charges which recede to infinity). With more complicated configurations of image charges (for example, with the number of such charges increasing with a, the second term can be reduced in magnitude and even eliminated entirely, giving

$$\frac{1}{\nabla^2} f(x) = \frac{1}{2} \int dt^1 \mid x^1 - t^1 \mid f(x^0, t^1). \tag{5.13}$$

The Wightman functions (5.10) with (5.11) and (5.13) correspond to Brown's [2] set of Green's functions for QED in Coulomb gauge (except for a finite renormalization). They can also be obtained using our original limiting procedure, but with $\zeta^{\pm}(x^0)$ in (5.6) taken to satisfy

$$[\zeta_a^-(x^0), \zeta_a^+(x^0)] = -\frac{\pi}{2} \int dt^1\, \partial_0^2 \Delta^-(x^0 - y^0, t^1) \left(a - \frac{1}{2} \mid t^1 \mid\right) \tag{5.14}$$

instead of (5.7) and (5.8). The different choices of ∇^{-2} correspond to different gauges (all satisfying $A_1 = 0$), and are equally acceptable as Coulomb gauge solutions of QED (see [4]).

Let us now examine some of the properties of the Wightman functions (5.10). It suffices to look at the long distance behavior of the two-point function

$$\langle 0 \mid \phi(x)\, \phi^*(y) \mid 0 \rangle \to \exp\left[-\frac{m}{4} \mid x^1 - y^1 \mid e^{im(x^0 - y^0)}\right] W_0(x, y) \tag{5.15}$$
$$\mid x^1 - y^1 \mid \to \infty$$

to see that the Schwartz inequality is violated and one has a very pathological indefinite metric. Due to their exponential growth for space-like separations, the Wightman functions of the Coulomb gauge field ϕ_c are not tempered distributions. The test functions must be analytic in momentum space, and thus one cannot define the energy-momentum spectrum of the theory.

It should be remarked that since the above pathologies occur through application of a gauge transformation, as far as observables are concerned, one has the same picture as in the previous section. We are naturally led to representations corresponding to charge sectors by considering expectation values

$$\frac{\langle 0 \mid \phi_c(f)\, A \phi_c^*(f) \mid 0 \rangle}{\langle 0 \mid \phi_c(f)\, \phi_c^*(f) \mid 0 \rangle}, \qquad f \in \mathscr{D}. \tag{5.16}$$

Contrary to what is expected in the case of a gauge transformation of the first kind [6, 7], there exists no local charge-raising operator in the theory. In addition, in two dimensions the growth of the Coulomb potential for spacelike separations prevents the existence of any charge-raising operator in a positive definite Hilbert space. This is the reason for our troubles with the electron field in the Coulomb gauge, since ϕ_c is directly a charge-raising operator.

VI. The Vector Meson Theory

We shall investigate in this section the way in which quantum electrodynamics can be understood as a limit of a vector meson theory when the bare mass of the meson tends to zero [8].

Starting from the coupled equations

$$i\gamma^\mu \partial_\mu \phi(x) + \frac{e}{2} \lim_{\substack{\epsilon\to 0\\ \epsilon^2<0}} \gamma^\mu [B_\mu(x+\epsilon)\, \phi(x) + \phi(x)\, B_\mu(x-\epsilon)] = 0, \tag{6.1}$$

$$\partial_\nu F^{\mu\nu}(x) + m_0{}^2 B^\mu(x) = -ej^\mu(x), \tag{6.2}$$

where $j^\mu(x)$ is the gauge-invariant current. (Since we are interested only in the electrodynamics limit we do not take the more general current definition of Hagen [4].)

Making the Ansatz

$$\phi(x) = e^{i\chi^+(x)} \psi(x)\, e^{i\chi^-(x)} \tag{6.3}$$

with

$$\begin{aligned} \chi^\pm(x) &= \alpha j^\pm(x) + \alpha(\sqrt{\pi} - \alpha)\, Q\Delta_\kappa{}^\pm(x) + \beta(\sqrt{\pi} - \alpha)\, \tilde{Q}\tilde{\Delta}_\kappa{}^\pm(x) \\ &\quad + \gamma^5 \Big(\beta \tilde{j}^\pm(x) + \frac{e}{m} \tilde{\Sigma}^\pm(x) \\ &\quad + \alpha(\sqrt{\pi} - \beta)\, \tilde{Q}\Delta_\kappa{}^\pm(x) + \beta(\sqrt{\pi} - \beta)\, Q\tilde{\Delta}_\kappa{}^\pm(x)\Big), \end{aligned} \tag{6.4}$$

$\phi(x)$ transforms as a "spin $\frac{1}{2}$" field and satisfies local anticommutation relations as long as [5]

$$(\alpha - \sqrt{\pi})(\beta - \sqrt{\pi}) = \pi. \tag{6.5}$$

Inserting (6.3) into (6.1), we get

$$\begin{aligned} B^\mu(x) &= (\alpha - \beta) \frac{\sqrt{\pi}}{e} \Big(j_f{}^\mu(x) - \frac{\alpha}{\sqrt{\pi}} \partial^\mu \Delta_\kappa(x)\, Q - \frac{\beta}{\sqrt{\pi}} \partial^\mu \tilde{\Delta}_\kappa(x)\, \tilde{Q}\Big) \\ &\quad - \frac{1}{m} \epsilon^{\mu\nu} \partial_\nu \tilde{\Sigma}(x). \end{aligned} \tag{6.6}$$

The gauge-invariant current defined as in electrodynamics (4.7) leads to

$$j^\mu(x) = \Big(1 - \frac{\beta}{\sqrt{\pi}}\Big)\Big(j_f{}^\mu - \frac{\alpha}{\sqrt{\pi}} \partial^\mu \Delta_\kappa(x)\, Q - \frac{\beta}{\sqrt{\pi}} \partial^\mu \tilde{\Delta}_\kappa(x)\, \tilde{Q}\Big) - \frac{e}{m\pi} \epsilon^{\mu\nu} \partial_\nu \tilde{\Sigma}(x), \tag{6.7}$$

where the first three terms on the r.h.s. of (6.7) correspond to the free current with infrared corrections [5].

With (6.2) we get

$$\Box\tilde{\Sigma}(x) + \left(m_0^2 + \frac{e^2}{\pi}\right)\tilde{\Sigma}(x) = 0, \tag{6.8}$$

and

$$\left[m_0^2(\alpha - \beta)\sqrt{\pi} + \left(1 - \frac{\beta}{\sqrt{\pi}}\right)e^2\right] \times \left[j_f^\mu - \frac{\alpha}{\sqrt{\pi}}\,\partial^\mu \Delta_\kappa(x)\,Q - \frac{\beta}{\sqrt{\pi}}\,\partial^\mu\tilde{\Delta}_\kappa(x)\,\tilde{Q}\right] = 0, \tag{6.9}$$

leading to

$$m^2 = m_0^2 + \frac{e^2}{\pi}, \tag{6.10}$$

and, with (6.5) to insure locality when $m_0^2 \neq 0$,

$$\beta = \sqrt{\pi}\left[1 \pm \left(\frac{\pi m_0^2}{e^2 + \pi m_0^2}\right)^{1/2}\right], \qquad \alpha = \sqrt{\pi}\left[1 \pm \left(\frac{e^2 + \pi m_0^2}{\pi m_0^2}\right)^{1/2}\right]. \tag{6.11}$$

We see that as $m_0 \to 0$, $\beta \to \sqrt{\pi}$ and $\alpha \to \infty$. The Wightman functions of the fermion fields are

$$\langle 0 \mid \phi(x_1) \cdots \phi(x_n)\,\bar{\phi}(y_1) \cdots \bar{\phi}(y_n) \mid 0\rangle = e^{iG(x,y)}\,W_0(x, y), \tag{6.12}$$

with

$$G(x, y) = \sum_{j<k}\Big\{(a + b\gamma^5_{x_j}\gamma^5_{x_k})\,D^-(x_j - x_k) + (a + b\gamma^5_{y_j}\gamma^5_{y_k})\,D^-(y_j - y_k) + \left(\frac{e}{m}\right)^2 (\gamma^5_{x_j}\gamma^5_{x_k}\Delta^-(x_j - x_k) + \gamma^5_{y_j}\gamma^5_{y_k}\Delta^-(y_j - y_k))\Big\} + \sum_{j,k}\Big\{(-a + b\gamma^5_{x_j}\gamma^5_{y_k})\,D^-(x_j - y_k) + \left(\frac{e}{m}\right)^2 \gamma^5_{x_j}\gamma^5_{y_k}\Delta^-(x_j - y_k)\Big\}. \tag{6.13}$$

Taking the limit $m_0^2 \to 0$ one sees that fermion Wightman functions diverge. The divergent part can, however, be gauged away, in two different ways:

1. Instead of ϕ given by (6.3) one can consider $e^{i\alpha\eta^+(x)}\phi(x)\,e^{i\alpha\eta^-(x)}$ with η an independent zero mass free field with indefinite metric, which in the limit of

$m_0^2 \to 0$ leads us to Schwinger's solution. The reason that one does not in this case satisfy Maxwell's equations on the whole Hilbert space (cf. Section II) is that although formally the free part of the current (6.7) vanishes as $m_0^2 \to 0$, its commutator with the fermion fields tends to a nonzero limit (cf. (2.15)).

2. Making a nonlocal gauge transformation

$$\phi(x) \to e^{i\alpha' j^+(x)} \phi(x)\, e^{-i\alpha' j^-(x)}$$

with the appropriate infrared corrections, and taking $\alpha' - \alpha$ finite as $m_0^2 \to 0$, one is led to the noncovariant solutions discussed in Section III.

VII. Broken Symmetry Aspects

The equations of motion of quantum electrodynamics in two dimensions with zero electron mass are invariant under both gauge and γ^5 transformations of the first kind. It is worthwhile to examine the spontaneous breaking of these symmetries in various operator solutions of Sections II and III.

In the covariant solution of Section II the gauge-invariant current (2.11) generates local gauge transformations and one also has a conserved current

$$j^{5\mu}(x) = \epsilon^{\mu\nu} \left(j_\nu(x) - \frac{m}{\sqrt{\pi}} A_\nu(x) \right) = j^{5\mu}_{\text{free}}(x)$$

generating local γ^5 transformations. The latter current, however, is not gauge invariant, since it does not commute with $j_L{}^\mu$ (cf. [15, 16]). There is no spontaneous breaking of the symmetries.

In the noncovariant gauges of Section III there is no local current generating the γ^5 transformations, and for $\alpha = \sqrt{\pi}$ there is also no current generating the gauge transformations. This corresponds to a total breakdown of Noether's theorem, a mechanism proposed by Maris *et al.* [17] as a way to avoid Goldstone bosons [18] when a symmetry is spontaneously broken. In fact, we have a spontaneous breakdown of both symmetries (cf. (4.14), (4.17)) without zero mass particles. The absence of conserved currents implies that there are no local automorphisms of the field algebra corresponding to the symmetries, and thus Goldstone's theorem [18–22] fails (cf. [12] for a similar effect in superconductivity).

For $\alpha \neq \sqrt{\pi}$ a local current generating gauge transformations of the first kind does exist and is proportional to the vector potential (3.8), i.e.,

$$[A^{\alpha 0}(y), \phi^\alpha(x)]_{\text{E.T.}} = \frac{\sqrt{\pi}}{m} \left(\frac{\alpha}{\sqrt{\pi}} - 1 \right)^2 \phi^\alpha(x)\, \delta(x^1 - y^1) + \text{Schwinger terms.}$$

Although $A^{\alpha\mu}$ creates zero-mass gauge excitations for $\alpha \neq \sqrt{\pi}$, these do not correspond to Goldstone bosons, since $\phi^\alpha(x_1) \cdots \phi^\alpha(x_n)$ (for example) creates gauge excitations with a continuous mass spectrum near the origin (cf. (3.13) and [5]), and, therefore, gauge invariance is not spontaneously broken [22]. On the other hand, the γ^5 invariance, which is not locally generated by a conserved current, is spontaneously broken for any α, as can be seen from the failure of the linked-cluster property (cf. (3.13) and [5])

$$\lim_{a\to\infty} \langle 0 \mid \phi_1(x)\, \phi_2^*(y)\, \phi_2(y^0, y^1 + a)\, \phi_1^*(x^0, x^1 + a) \mid 0\rangle$$

$$\neq |\langle 0 \mid \phi_1(x)\, \phi_2^*(y) \mid 0\rangle|^2 = 0$$

leading to irreducible representations without γ^5 invariance.

To summarize, we see that the breaking or not of the gauge and γ^5 symmetries depends on the q-number gauge employed. However, in none of the solutions studied do we find Goldstone bosons. In every case of spontaneous symmetry-breaking, there is no corresponding conserved generator of the local transformations, and hence Goldstone's theorem is not applicable.

The question arises whether any observable significance can be attributed to the broken symmetries of two-dimensional quantum electrodynamics. The answer seems to be yes: the absence of charge (and pseudocharge) quantization (i.e. of *discrete* charge sectors) among the irreducible representations of the observable algebra is a gauge-invariant expression of spontaneous symmetry breaking. This *physical* breakdown of gauge and γ^5 invariance, an immediate consequence of the lack of physical electron spectrum in the theory, leads directly to spontaneous breakdown in the usual, *mathematical* sense in the noncovariant representation with $\alpha = \sqrt{\pi}$. In other gauges, the simple structure of the algebra of observables, and along with it the physical breakdown of symmetry, may be masked by the presence of spurious gauge excitations. In such gauges, whether or not the symmetries are broken in the usual sense is physically of no importance.

Appendix

We wish to obtain here the connection between the renormalized fields and Wightman functions which we employed throughout this work and the unrenormalized ones which are used by many authors [1–4]. The generic form of our Wightman functions is

$$\langle 0 \mid \phi(x_1) \cdots \phi(x_n)\, \bar{\phi}(y_1) \cdots \bar{\phi}(y_n) \mid 0\rangle = e^{iF(x,y)} W_0(x, y) \qquad \text{(A.1)}$$

with

$$iF(x, y) = \sum_{j<k} \{A(x_j - x_k) + \gamma^5_{x_j}\gamma^5_{x_k} B(x_j - x_k) + (\gamma^5_{x_j} + \gamma^5_{x_k})\, C(x_j - x_k)$$
$$+ A(y_j - y_k) + \gamma^5_{y_j}\gamma^5_{y_k} B(y_j - y_k) - (\gamma^5_{y_j} + \gamma^5_{y_k})\, C(y_j - y_k)\}$$
$$+ \sum_{j,k} \{-A(x_j - y_k) + \gamma^5_{x_j}\gamma^5_{y_k} B(x_j - y_k) + (\gamma^5_{y_k} - \gamma^5_{x_j})\, C(x_j - y_k)\},$$

where A, B, C are sums or differences of free two-point functions.

Because of parity invariance $C(x_1) = -C(-x_1)$, so $C(0) \overset{\text{def}}{=} 0$.

$A(0)$ and $B(0)$ are in general infinite but can be finite and real in special gauges. In those cases we can go to the unrenormalized fields and Wightman functions by writing

$$iF(x, y) = i\hat{F}(x, y) - nA(0) + \left\{ \sum_{j<k} (\gamma^5_{x_j}\gamma^5_{x_k} + \gamma^5_{y_j}\gamma^5_{y_k}) + \sum_{j,k} \gamma^5_{x_j}\gamma^5_{y_k} \right\} B(0), \quad \text{(A.3)}$$

where $\hat{F}$ is obtained from F by replacing A, B, C by $A - A(0)$, $B - B(0)$, $C - C(0)$.

Therefore

$$\exp[iF(x, y)]\; W_0(x, y)$$
$$= \exp\left[i\hat{F}(x, y) - nA(0) - nB(0) + \frac{1}{2}\left(\sum_j (\gamma^5_{x_j} + \gamma^5_{y_j}) \right)^2 \right] W_0(x, y). \quad \text{(A.4)}$$

The γ^5 invariance of the free Wightman functions implies

$$\exp\left(i\lambda \sum_j (\gamma^5_{x_j} + \gamma^5_{y_j}) \right) W_0(x, y) = W_0(x, y), \quad \text{(A.5)}$$

so

$$\exp\left\{ \frac{1}{2} \left[\sum_j (\gamma^5_{x_j} + \gamma^5_{y_j}) \right]^2 \right\} W_0(x, y) = W_0(x, y), \quad \text{(A.6)}$$

and, with (A.4),

$$e^{iF(x,y)} W_0(x, y) = e^{i\hat{F}(x,y)} e^{-n(A(0)+B(0))} W_0(x, y). \quad \text{(A.7)}$$

Thus, defining the unrenormalized fields by

$$\hat{\phi}(x) = e^{1/2(A(0)+B(0))} \phi(x), \quad \text{(A.8)}$$

we are led to the unrenormalized Wightman functions

$$\langle 0 \mid \hat{\phi}(x_1) \cdots \hat{\phi}(x_n)\, \bar{\hat{\phi}}(y_1) \cdots \bar{\hat{\phi}}(y_n) \mid 0 \rangle = e^{i\hat{F}(x,y)} W_0(x, y). \quad \text{(A.9)}$$

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